Salud, Ciencia y Tecnología. 2025; 5:1833
doi: 10.56294/saludcyt20251833

ORIGINAL

Analysis of students' errors in drawing function graphs in three-dimensional space based on APOS theory

Análisis de los errores de los alumnos al dibujar gráficas de funciones en el espacio tridimensional basado en la teoría APOS

Yerizon Yerizon[1], Made Arnawa[2], Arnellis Arnellis[1], Fridgo Tasman[1], Yarman UNP[1], Kelly Angelly Hevardani[1]

[1]Department of Mathematics, Universitas Negeri Padang. Padang, Indonesia.
[2]Department of Mathematics, Universitas Andalas. Padang, Indonesia.

Cite as: Yerizon Y, Arnawa M, Arnellis A, Tasman F, UNP Y, Hevardani KA. Analysis of students' errors in drawing function graphs in three-dimensional space based on APOS theory. Salud, Ciencia y Tecnología. 2025; 5:1833. https://doi.org/10.56294/saludcyt20251833

Submitted: 11-01-2025 **Revised:** 28-03-2025 **Accepted:** 27-06-2025 **Published:** 28-06-2025

Editor: Prof. Dr. William Castillo-González

Corresponding Author: Yerizon Yerizon

ABSTRACT

This research is aimed to investigate students' conceptual understanding of drawing graphs in three-dimensional space through the application of APOS theory. Additionally, genetic decomposition is designed for graphs to facilitate conceptual understanding and assist lecturers in planning learning activities. A qualitative descriptive method is also adopted and data collection is performed through tests, interviews, and documentation, while the validity techniques use triangulation. The results show that students with high, moderate, and low abilities have an understanding of the action, process, and object stages in drawing graphs in three-dimensional space. However, those with moderate and low abilities lack understanding at the object stage. Some reported errors are challenges in creating graphs, with instances of being erroneously constructed in two-dimensional space. There are also errors in determining the intersection point of lines with Cartesian coordinate axis, as well as in drawing and connecting traces.

Keywords: Errors; Genetic Decomposition; APOS Theory; Graphics; Three-Dimensional Space.

RESUMEN

Esta investigación tiene como objetivo investigar la comprensión conceptual de los estudiantes sobre el dibujo de gráficos en el espacio tridimensional mediante la aplicación de la teoría APOS. Además, se diseña una descomposición genética de los gráficos para facilitar la comprensión conceptual y ayudar a los profesores a planificar las actividades de aprendizaje. También se adopta un método cualitativo descriptivo y la recogida de datos se realiza mediante pruebas, entrevistas y documentación, mientras que las técnicas de validez utilizan la triangulación. Los resultados muestran que los alumnos con capacidades altas, moderadas y bajas comprenden las etapas de acción, proceso y objeto en el dibujo de gráficos en el espacio tridimensional. Sin embargo, aquellos con capacidades moderadas y bajas carecen de comprensión en la etapa de objeto. Algunos errores reportados son desafíos en la creación de gráficos, con casos de ser erróneamente construidos en el espacio bidimensional. También hay errores en la determinación del punto de intersección de líneas con ejes de coordenadas cartesianas, así como en el trazado y conexión de trazos.

Palabras clave: Errores; Descomposición Genética; Teoría APOS; Gráficos; Espacio Tridimensional.

INTRODUCTION

Students are experiencing challenges in mathematics and science courses.[1] Calculus is a prerequisite for further courses and one of the items on the general intelligence test.[2,3] In the calculus course during the initial semester, students engage in the analysis of graphs within three-dimensional space. This academic pursuit is followed by the exploration of Multivariable Calculus (MC) in the third semester. Within the curriculum, proficiency is acquired in comprehending functions including two or more variables within two or three-dimensional space. The intended learning outcomes comprise the development of ability to comprehend and apply concepts related to coordinate systems, partial derivatives, double integrals, and practical applications, with an understanding of infinite series. The MC lectures commence with an exposition on the Cartesian coordinate system within the plane (R2), progressing to the polar coordinate system.

Students are introduced to the interrelation between the Cartesian and polar coordinate systems while acquiring skills to solve equations. The instructional sessions proceed to address the coordinate system in three-dimensional space (R3), commencing with the mapping of points across various octants. Students are educated on the general forms and techniques for graphing flat planes, cylinders, and elliptic paraboloids within three-dimensional space. However, there is a high struggle to understand the concepts, and often make errors in drawing graphs. Previous results have confirmed a lack of conceptual understanding regarding graphs in three-dimensional space. It will hinder students' ability to transfer and generalize knowledge.[4] As stated by Aniswita et al.[5], an individual is considered to have a great grasp of mathematics when she can make connections between each of them of knowledge. Therefore, the mental constructions of mathematical knowledge facilitating individuals' understanding should be investigated. This research relies on a constructivist paradigm in learning, specifically APOS (Action-Process-Object-Schema) theory, where mental constructions leading to the understanding of mathematical concepts can be modeled. APOS theory provides a model for learning mathematical concepts and analyzing the construction of individuals' knowledge.[6]

This theory can describe the formation of mathematical knowledge within individuals. The objective to be achieved is the formation of students' mental constructions.[7] Action is an externally driven transformation of previously conceptualized mathematical objects. This is performed by individuals in response to external stimuli and is carried out explicitly, step by step, following instructions. Individuals are less reliant on external signals and perform transformation with internal operations when internalizing an action into a process. For individuals to apply an action to a process, the process should be understood where transformations can be operated and constructed. At this stage, individuals have encapsulated the process into an object. Besides the mechanism, a new object can also be formed by de-encapsulating one or more previously formed objects into the process. A schema is a collection of coherent mental structures and the relationships between these structures used by individuals in facing mathematical problems.[6]

Students with conceptual understanding also have good procedural knowledge.[8] Gaisman et al.[9] investigated the relationship between ideas on subsets of three-dimensional space and understanding of function graphs with two variables. Understanding can be associated with the schema structure in R3 and flexibility in using different representations. Most students struggle to comprehend function graphs with two variables following the inability to build a well-developed schema for R3. On the contrary, Martínez-Planell et al.[10] found that significant difficulties were not experienced in creating graphs and describing the functions of two variables. Most of the constructions predicted were performed by genetic decomposition because the coordination of sets, functions of one variable, and R3 schema played a role in creating function graphs with two variables.

Previous results used theoretical frameworks in investigating mathematical understanding. Meanwhile, this research uses APOS theory developed by Dubinsky in the 1980s.[6] Reflective abstraction as the construction of mathematical logic structures by individuals during the cognitive development process. Dubinsky[11] used the approach to identify students' difficulties in understanding mathematical concepts. Furthermore, the concepts that have been the focus of APOS-based research include function limits,[12] derivatives,[13] and linear algebra.[14] Maharaj[12] used APOS theory to investigate the understanding of function limits. Students experienced challenges in understanding the concept of limits, primarily attributed to a deficiency in the appropriate mental structures at the levels of process, object, and schema.

Oktaç et al.[15] analyzed written answers based on APOS theory and presented conclusions regarding mental constructions of linear transformations. This was followed by interviews with some participants to investigate the level of understanding. In addition, Siyepu[16] explored students' errors in solving function derivative questions. Based on the results, there is a limited association between mental constructions and the types of errors. The use of APOS theory for data analysis is examined by connecting the error classification in written answers with mental structures. Finding research on students' mental frameworks when drawing three-dimensional graphs is uncommon, yet. Studying students' mental frameworks and errors that they produce when constructing three-dimensional graphs serves as essential. The research questions are: What does genetic decomposition look like for drawing three-dimensional graphs?; How are student answers analyzed?; and What are the mistakes students make in drawing three-dimensional graphs?

METHOD

Research Subject

This research comprises third-semester students of the mathematics study program at the FMIPA Universitas Negeri Padang. This study involved two classes, namely in class A there were 37 students and class B there were 36 students. All of these students took the multivariable calculus course in the July-December 2023 semester. The understanding level of students is divided into three categories, namely high, moderate, and low ability. Based on this, one student each was randomly selected to represent the three abilities.

Instruments

The investigation into conceptual understanding based on APOS theory begins with the design of the Genetic Decomposition (GD) for the concept. GD is defined as a set of mental constructs that depict the thought processes of students in understanding a concept. This is designed by considering the epistemological nature of the concept, the mathematical understanding, knowledge of APOS theory, and experience in teaching.(10) Each individual has a different thought process in constructing the understanding.(6) This research designs a GD in drawing graphs in three-dimensional space, as shown in figure 1.

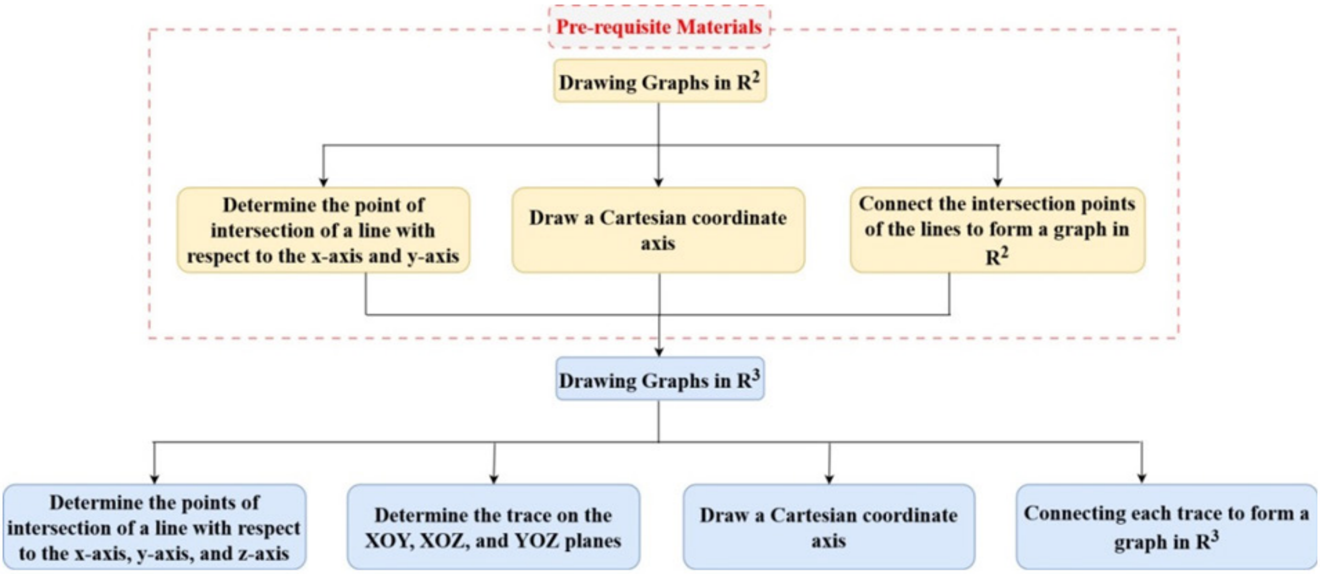


Figure 1. GD in drawing graphs in three-dimensional space

Based on figure 1, students should master prerequisite material before drawing graphs in three-dimensional space. Graphs are first drawn in two-dimensional space, which serves as prerequisite material, starting from determining the intersection points of lines with x and y axes. Subsequently, Cartesian coordinate axes are drawn and connected with the intersection points. The concept of drawing graphs in three-dimensional space is easily understood after students have become proficient in drawing graphs in two-dimensional space. Initially, XOY, XOZ, and YOZ planes should be determined before drawing and connecting the traces accurately. Furthermore, interview guidelines are used to dig deeper into information related to answers. The results show the thought process of students in solving the question. Draw the graph of $z - 9 + x^2 + y^2 = 0$ in three-dimensional space!

Data Collection and Analysis

Table 1. Classification of students' errors in drawing graphs in three-dimensional space

Students' Answers	Error Classification
Not making graphs. Even when there is, the graph is made in two-dimensional space.	Arbitrary Error (AE)
Incorrect in determining the point of intersection of the line with the Cartesian coordinate axis and the trace on each plane.	Structural Error (SE)
Incorrect in drawing and connecting the traces on each plane.	Executive Error (EE)

Students work on two questions related to graphs in three-dimensional space. Furthermore, the answers written by students are accessed with a weight of 10. The answers are categorized as correct when the students succeed in determining the intersection point of the line with x, y, and z axes, determining the traces on each plane, drawing the coordinate axes, and connecting the traces to form a graph in three-dimensional space. However, the inability to draw and connect the traces on the Cartesian coordinate axes will be classified as a partially correct answer. Students' answers are analyzed using the error classification according to Orton(17), as shown in table 1. Besides analyzing answers, this research also conducts interviews with three students who have different abilities. The aim is to explore information related to the provided answers and find out the thinking patterns of students in drawing graphs.

RESULTS

The achievement of students in painting graphs in three-dimensional space can be seen in figure 2. 56 % reached the action and process stages while 44 % reached the action, process, and object stages in drawing graphs. Out of the 44 %, 25 % made mistakes and 19 % managed to draw graphs in three-dimensional space correctly. The data is obtained from students' achievements in each mental construct of APOS theory, as shown in table 2.

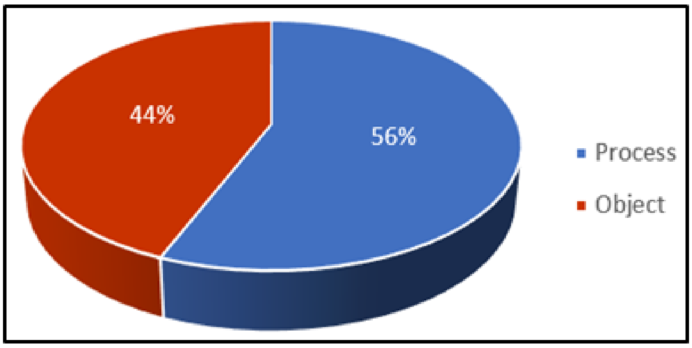


Figure 2. Achievement of APOS stages in question

Table 2. Indicators of students' concept understanding based on APOS theory

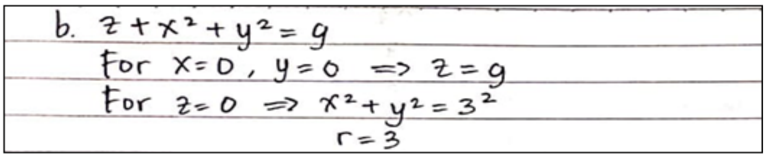
Topic	Stages of APOS	Indicator
Graphs in Three-dimensional Space	Action	Students are able to: Identify what is known and asked in the question.
	Process	Students are able to: Determine the points of intersection of the line concerning the *x*, *y*, and *z* axes. Determine the coordinates of the vertex of the parabola.
	Object	Students are able to: Draw a Cartesian coordinate axes. Draw traces on the plane to form a graph in three-dimensional space. Describe the information needed to draw a graph in three-dimensional space.
	Schema	Students are able to: Use concepts and procedures in solving story questions related to graphs in three-dimensional space.

Students' errors in drawing graphs are also analyzed, as shown in table 3. There are no AE since students pay attention to the limits set in the question. Meanwhile, about 20,27 % and 59,46 % make SE and EE. From the questions above, the dominant errors in drawing graphs are EE.

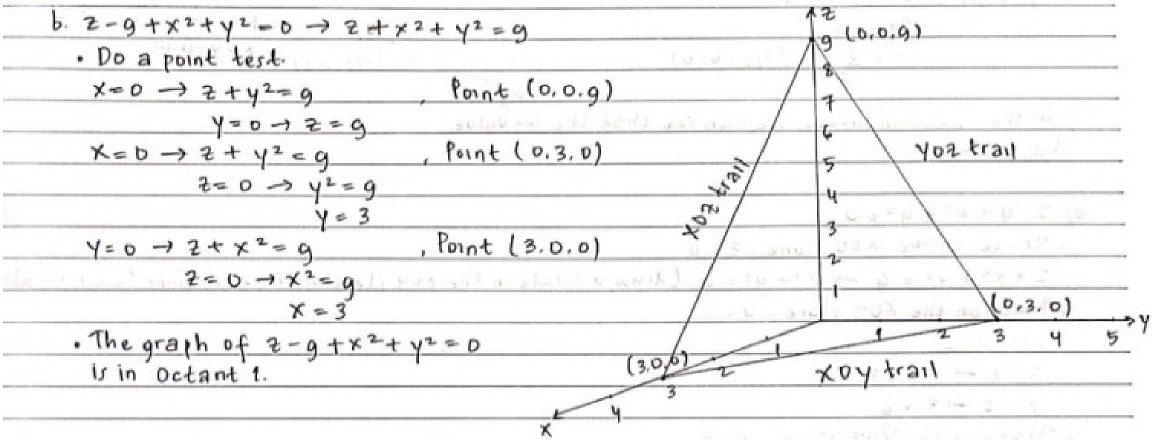
Table 3. Students' errors in painting graphs in three-dimensional space

Error Classification	Question
AE	-
SE	20,27 %
EE	59,46 %

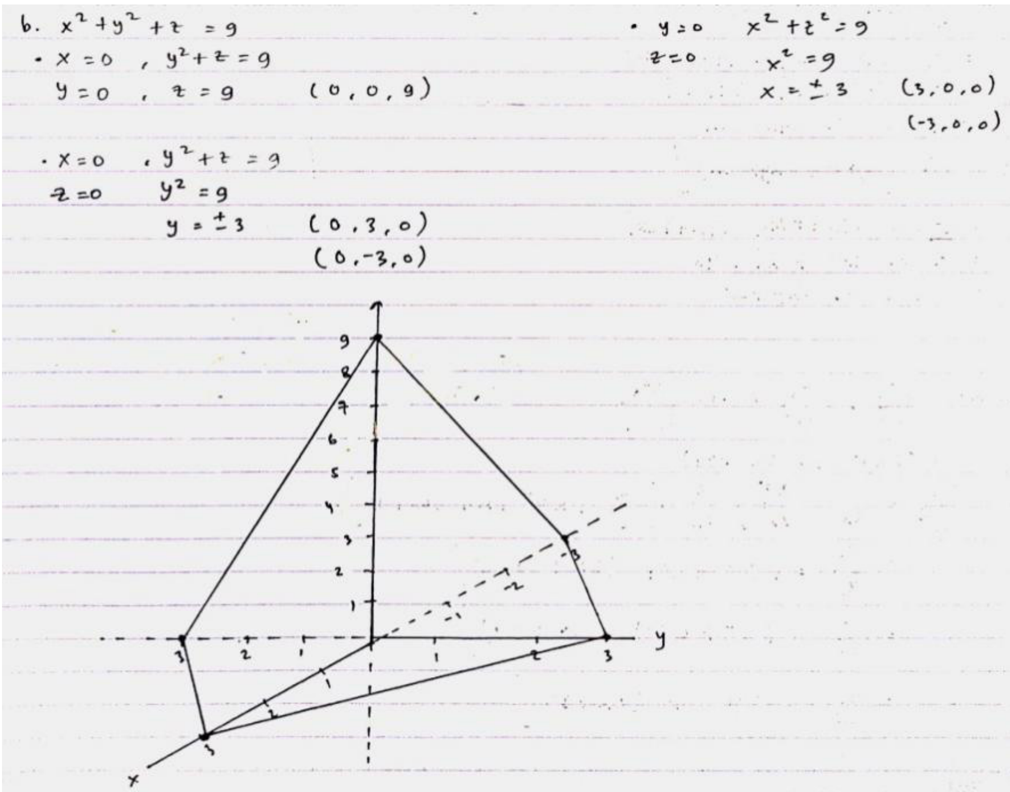
Students determine the intersection point of the line with Cartesian coordinate axis but cannot draw the graph in three-dimensional space (AE), as shown figure 3a. Errors are made in determining the intersection point with the Cartesian coordinate axis and the trace on each plane (SE) since the graph formed is wrong, as shown figure 3b. In contrast to figure 3c, there is success in determining the intersection point of the line with Cartesian coordinate axis. However, students fail to draw and connect the traces on each plane (EE).



(a)



(b)



(c)

Figure 3. AE (a), SE (b), and EE (c)

High-ability student (HS)

In figure 4, HS writes the known equation and adds 9 to the two segments to obtain equation $z + x^2 + y^2 = 9$. XOY, XOZ, and YOZ planes are determined before drawing the traces on Cartesian coordinates. To further explore HS's understanding in solving question, the following interview is conducted.

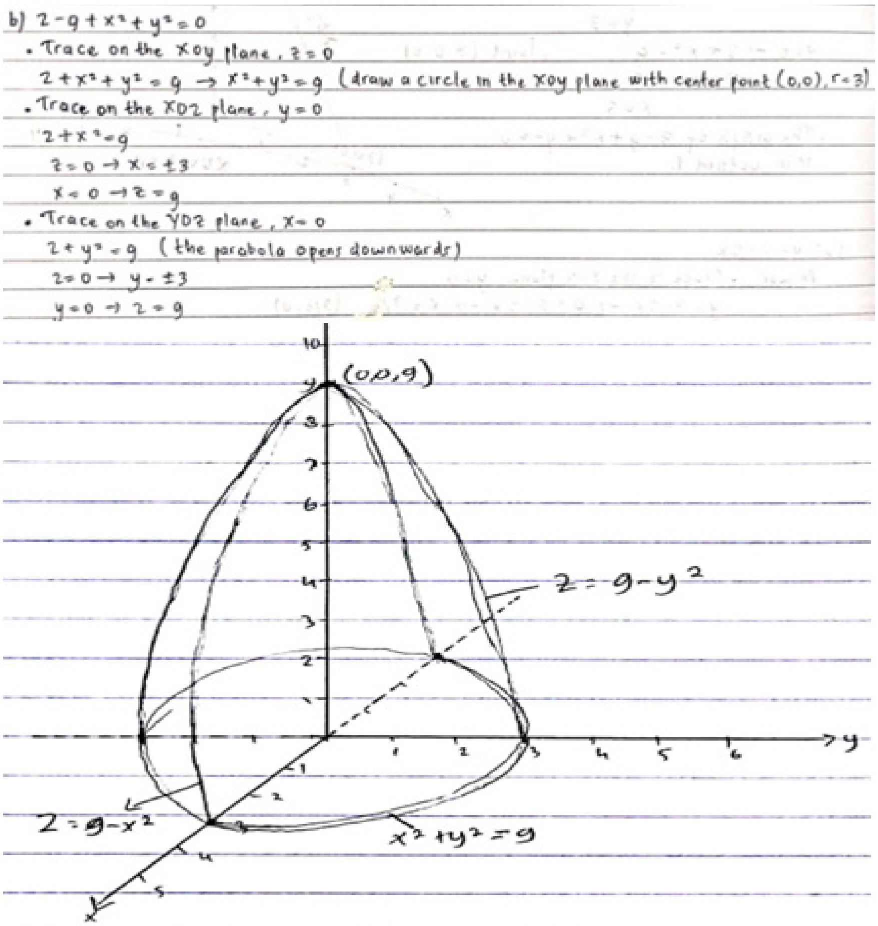


Figure 4. HS's answer

HS can identify what is known and asked in the question, determine the coordinates of parabola vertex, draw Cartesian coordinate axis, as well as obtain traces on the plane. Therefore, there is an understanding of the action, process, and object stages in drawing graphs in three-dimensional space.

Medium-ability student (HS)

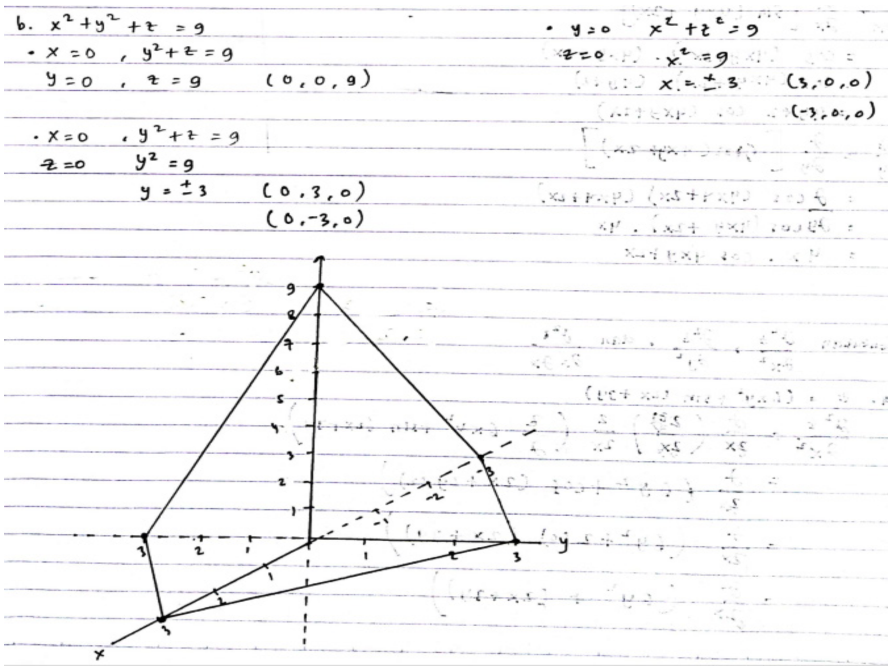


Figure 5. MS's answer

In figure 5, MS writes the known equation into $x^2 + y^2 + z = 9$. The intersection points of the line are determined concerning x, y, and z axes before connecting the points on Cartesian coordinate axis. Moreover, MS was interviewed by the researcher.

Researcher: try to explain the trace on each field that you get!

MS: (silence)

Researcher: if we want to determine the trace on the XOY plane, what variable is 0?

MS: z = 0, means the trace is x2 + y2 = 9 ma'am.

Researcher: okay, good. How about the traces on the XOZ and YOZ planes?

MS: if we look at the trace on the XOZ plane, it means y = 0, hence we get z + x2 = 9. For the trace on the YOZ plane, it means x = 0, hence we get z + y2 = 9.

Researcher: how do you connect these traces to form a graph in three-dimensional space?

MS: I made a mistake in drawing the graph.

Researcher: why is that?

MS: because the trace on the XOY plane should form a circle with the center at (0, 0) and a radius (r) of 3. Yesterday, I only connected each point obtained with a straight line.

Based on the interview above, MS can identify what is known and asked in the question, determine the vertex of a parabola, draw Cartesian coordinate axes, and obtain traces on each plane correctly. However, there is no possibility of drawing each of the traces on Cartesian coordinate axes. MS makes EE due to the failure in manipulation when associated with the classification of Orton(17). The graph created is incorrect since only the obtained points are connected with a straight line.

Low-ability student (LS)

Figure 6 shows that LS writes down the known equation as $x^2 + y^2 = 9 - z$. Subsequently, the vertex of the parabola is determined by substituting x = 0 and y = 0 into the equation $x^2 + y^2 = 9 - z$. Cartesian coordinate axes created are correct but LS draws an upward-opening parabola with the vertex (0, 0, 9), making the answer incorrect.

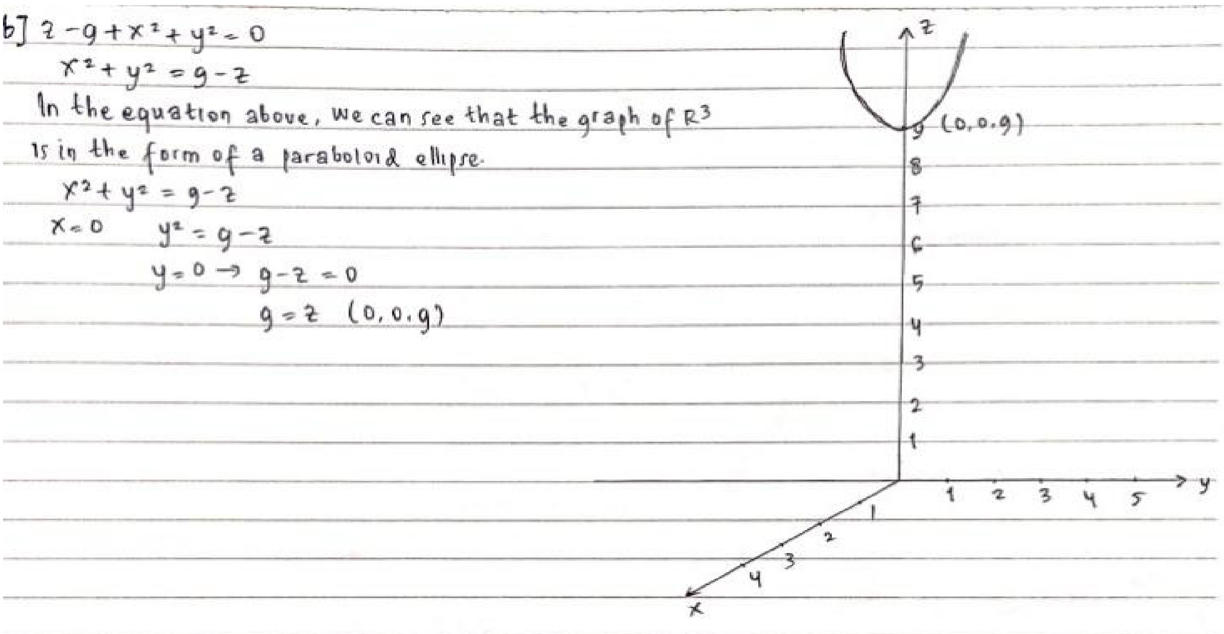


Figure 6. LS's answer

Researcher: what steps did you take to solve the question?

LS: first, we determine the values of x, y, and z. In this question, the graph is an upward-opening parabola with the vertex (0, 0, 9), ma'am.

Researcher: can you explain the traces on each plane that you obtained?

LS: i did not determine the traces on each plane, ma'am.

Researcher: why is that?

LS: i thought drawing the graph was enough by just knowing its vertex, ma'am.

Researcher: if we want to determine the trace on the XOY plane, which variable has a value of 0?

LS: the variable z, ma'am. Hence, the trace is a circle, right?

Researcher: okay, what about the traces on the XOZ and YOZ planes?
LS: we substitute x = 0, ma'am. (answering hesitantly)
Researcher: if we substitute x = 0, that means we are looking for the trace on which plane?
LS: (silent)
Researcher: how do you create the graph in three-dimensional space?
LS: from the point (0, 0, 9), we draw an upward-opening parabola, ma'am.

Based on the interview above, LS can identify what is known and asked in the question, determine the vertex of a parabola, and draw the Cartesian coordinate axes accurately. However, there is no possibility of determining and drawing traces on the Cartesian coordinate axes. SE is made due to the failure to understand the fundamental principles for the solution.

Table 4. Achievements of HS, MS, and LS

	APOS Stages/ Indicators					
Subject	Action	Process		Object		
	1	1	2	1	2	3
HS	√	√	√	√	√	√
MS	√	√	√	√	-	√
LS	√	√	√	√	-	-

The summary of achievements for HS, MS, and LS in solving question is presented in table 4. This question has 1, 2, and 3 indicators at the action, process, and object stages, respectively. HS achieves all indicators at the action, process, and object stages. MS and LS achieve all indicators at the action and process stages but only 2 and 1 at the object stage, respectively.

DISCUSSION

This research reports the use of APOS theory in investigating students' conceptual understanding of drawing graphs in three-dimensional space. The results show that HS, MS, and LS have an understanding of the action, process, and object stages in drawing graphs in three-dimensional space. However, the understanding of MS and LS is still lacking at the object stage. This is relevant to the findings of Yerizon et al.[18] where the understanding of MS at the object stage is not optimal and LS are still lacking at the process and object stages in constructing the concept of partial derivatives. Another finding showed that MS successfully reached the APOS stage when solving double integral using polar coordinates. On the other hand, LS failed to draw the integral area and determine the integral boundary correctly.[19]

Studying that fails to result in conceptual and procedural knowledge is the cause of students' conceptual ignorance.[5] Despite considering the mathematical concepts underlying the solutions, students purely resembled the methods used by their teachers.[20] According to previous study, HS are able to construct knowledge up to the object stage by accurately drawing graphs of linear equations and determining the points, MS can construct knowledge at the object stage but may experience slight errors in calculation algorithms, and LS cannot construct abilities at any stage. The following demonstrates that LS require extra assistance in comprehending mathematical concepts particularly any that tend to be abstract in character. Instructors must provide them with materials based on what they're seeking to acquire knowledge. Fauzan et al.[21] hence developed a mathematical cognition-based learning system for number teaching in order to instruct LS.

Math problems are attempted differently by each student. It seems evident from each student's capability, the complexity of the challenge, and how well the approach works to solve mathematical problems.[22] Students' ways of solving problems are influenced by their attitudes[23] and learning materials.[24] Contrary to Suliani et al.[25] findings that students' diverse beliefs don't affect their metacognitive abilities in carrying out problem solving procedures. Students understand the value of educational materials that can facilitate their education.[26] According to Harisman et al.[27] investigation, there was no discernible variation in the utilization of the serving backward approach between male and female students. Put another way, applying serving backwards techniques yields better and more efficient results while tackling high school mathematical issues. This is primarily a result of the strategy's quick processing and minimal number of calculating processes.

In the action stage, male students draw graphs directly on Cartesian plane and use the general form of quadratic functions, while the female counterparts create pairs of point tables to make graphs and use the general form.[28] In the process stage, male students substitute some numbers into the equation and then connect the points, while females determine intersections with the coordinate axes, find other points, and observe the direction of the graph. In the object stage, male students create graphs by shifting the top part of the graph, while females determine the function formula by adding the constant from the original graph and both genders have not reached the scheme stage.

Research on students' thinking levels has also been conducted in several countries on various topics. Martínez-Planell et al.[29] applied APOS-based two-variable function learning through three cycles. Students using APOS-based activities were able to visualize surfaces, basic planes, curves, and other mathematical processes. Dynamic representations were produced related to two-variable functions without having to perform explicit calculations. Students could build meaningful connections with other mathematical knowledge to coordinate different processes and justify the procedures used without relying on memorized facts. Therefore, different APOS research cycles can be used to improve students' mathematical understanding.

Bansilal et al.[30] explored pre-service teacher understanding of surjective and injective functions through real analysis courses at the University of South Africa. Most pre-service teachers reached the action stage and did not show the object stage of the concept. Students identified subjective or injective functions using certain rules but were unable to explain the associated reason. This is due to a lack of understanding of function and set, which are prerequisite materials. Şefik et al.[31] showed the understanding of three-dimensional space and two-variable functions. Only one student was able to construct the concept of three-dimensional space as an object in the genetic decomposition framework. Some could not connect the concepts of two-variable functions and three-dimensional space. Díaz-Berrios et al.[32] also used APOS theory to explore the understanding of exponential and logarithmic functions. Group A students built some intended processes, while Group B needed more time to prove some constructions due to a lack of class discussion activities.

The understanding of high school teachers regarding linear algebra. The conceptual framework combined with APOS theory played a crucial role in the design, implementation, and data analysis of the teaching experiment.[33] Some of the difficulties in solving linear algebra problems include a lack of attention to the use of concepts, ideas, and symbols in linear algebra, understanding the concept of linear algebra at the process, object, and scheme levels, and class participation. The concept of gradient can be formed by associating geometric comparisons, algebra, as well as function and linear constant properties. The features also considered are parametric coefficients, behavioral indicators, physical and determining properties, real-world situations, trigonometric concepts, and calculus.[34]

The understanding of the concept of limits improved after using GeoGebra integrated with a multi-teaching approach in APOS paradigm.[35] Martínez-Planell et al.[36] reported students' understanding of the relationship between Riemann sums and double integral. The object stage was not shown and this indicated the importance of each mental construction. The same topic was also examined by Borji et al.[37], which most students had not reached the process stage in visualizing the volume of solid objects and connecting Riemann sums with definite integrals. GD consists of cylinders, solid objects, cross-sectional area, Riemann sums, and geometric-analytic volume. Figueroa et al.[38] focused on scrutinizing the cognitive processes used by students when addressing pesticide modeling problems. Using APOS theory as a thinking framework, García-Martínez et al.[39] presented a cognitive model of mathematical induction principles in college. The use of mathematical induction principles based on APOS theory can identify difficulties and strategies used when engaging in activities with natural numbers.

Additionally, our investigation revealed a variety of student mistakes made when drawing graphs in three dimensions. We applied Orton[17] mistake categorization in this instance. A common mistake made by students when sketching and linking traces on each plane is EE. Unlike our error classification, errors made in the topic of linear equations with two variables include interpretation errors–mistakes in transforming questions into mathematical meanings, as well as drawing necessary conclusions to answer the questions, conceptual errors-mistakes made in answering questions with a single and many solutions, procedural errors–these mistakes are usually experienced in concluding the problem, and technical errors–calculation errors are made including the division of integers in fractional form.[40] Tatira[41] also found the same, students faced difficulties in procedural ways. Students struggled with problems that required higher-level mental construction, namely the process and object.

With different thinking levels, teachers can understand the development of mathematical performance and analyze cognitive abilities.[6] Students show distinct learning trajectories characterized by the capacity to transition between processes and actions, as well as the ability to move between objects and processes, with the flexibility to revert from one to the other. In understanding mathematical concepts, each student has different characteristics, and the mental constructions are subjected to different stages.[41,42] The success or failure in solving mathematical problems could be seen from the mental constructions achieved.[7] APOS theory framework is an effective method for analyzing the construction and relationships of mathematical concepts. It can facilitate students in building ideas related to a mathematical concept.[34,43] As Arnawa et al.[44] found that students' level of understanding of elementary linear algebra based on APOS theory is better than the traditional approach.

CONCLUSIONS

To draw three-dimensional graphs, students must be proficient in drawing two-dimensional graphs. Firstly, students determine the intersection points with respect to the x, y, and z axes. Then proceed to determine the trace with respect to the XOY, XOZ, and YOZ planes. Students draw the Cartesian coordinate axes and connect each trace so that a three-dimensional graph is formed. High, moderate, and low-ability students were reported to possess an understanding of the action, process, and object levels in drawing graphs. However, the understanding of moderate and low-ability students at the object level was still lacking and errors were from a lack of attention to the specified limits. Errors were also made in determining the intersection points of lines with the coordinate axis and traces on each plane, as well as drawing and connecting traces on each plane. Therefore, future research should be conducted to refine the GD, modify related instruments, and expand the error classification in data analysis.

ACKNOWLEDGEMENTS

The authors would like to thank Lembaga Penelitian dan Pengabdian Masyarakat Universitas Negeri Padang for funding this work with a contract number: 1704/UN35.15/LT/2024 fiscal year 2024.

BIBLIOGRAPHIC REFERENCES

1. Naidoo J, Paideya V. Exploring the Possibility of Introducing Supplemental Instruction at Secondary School Level. South African J Educ. 2015;35(2):1-10.

2. Aziz S Al, Winanda RS. Differential Calculus Mind Map as Output Concept Analysis in Preliminary Research Phase. AIP Conference Proceedings. 2023. p. 1-5.

3. Yerizon, Arnawa IM, Fitriani N, Tajudin NM. Constructing Calculus Concepts through Worksheet Based Problem-based Learning Assisted by GeoGebra Software. HighTech Innov J. 2022;3(3):282-96.

4. Hurrell D. Conceptual Knowledge OR Procedural Knowledge or Conceptual Knowledge AND Procedural Knowledge: Why the Conjunction is Important to Teachers. Aust J Teach Educ. 2021;46(2):57-71.

5. Aniswita, Fauzan A, Armiati. Students' Conceptual and Procedural Knowledge on Integration: Reflections on Calculus Learning. AIP Conference Proceedings. 2023. p. 1-8.

6. Arnon I, Cottrill J, Dubinsky E, Oktac A, Fuentes SR, Trigueros M, et al. APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education. New York: Springer. 2014.

7. Afgani MW, Suryadi D, Dahlan JA. Analysis of Undergraduate Students' Mathematical Understanding Ability of the Limit of Function Based on APOS Theory Perspective. Journal of Physics: Conference Series. 2017. p. 1-7.

8. Mahir N. Conceptual and Procedural Performance of Undergraduate Students in Integration. Int J Math Educ Sci Technol. 2009;40(2):201-11.

9. Gaisman MT, Martínez-Planell R. Geometrical Representations in the Learning of Two-variable Functions. Educ Stud Math. 2010;73(1):3-19.

10. Martínez-Planell R, Gaisman MT. Graphs of Functions of Two Variables: Results from the Design of Instruction. Int J Math Educ Sci Technol. 2013;44(5):663-72.

11. Dubinsky E. Reflective Abstraction in Advanced Mathematical Thinking. Netherlands: Kluwer Academic Publishers. 1991.

12. Maharaj A. An APOS Analysis of Students' Understanding of the Concept of a Limit of a Function. Pythagoras. 2010;71:41-52.

13. Maharaj A. An APOS Analysis of Natural Science Students' Understanding of Derivatives. South African J Educ. 2013;33(1):1-19.

14. Ndlovu Z, Brijlall D. Pre-service Mathematics Teachers' Mental Constructions when Using Cramer's Rule. South African J Educ. 2019;39(1):1-13.

15. Oktaç A, Trigueros M, Romo A. APOS Theory: Connecting Research and Teaching. Learn Math. 2019;39(1):33-7.

16. Siyepu SW. Analysis of Errors in Derivatives of Trigonometric Functions. Int J STEM Educ [Internet]. 2015;2:1–16.

17. Orton A. Students' Understanding of Integration. Educ Stud Math. 1983;14:1–18.

18. Yerizon, Sukestiyarno, Arnellis, Suherman, Hevardani KA. Analysis of Students' Mental Construction in Understanding the Concept of Partial Derivatives Based on Action-Process-Object-Schema Theory. Nurture. 2024;18(4):735–49.

19. Yarman, Dwina F, Murni D, Yerizon, Hevardani KA. Analysis of Concept Construction and Student Errors on the Topic of Double Integral Based on APOS Theory. J Pendidik Mat. 2024;18(3):367–86.

20. Tasman F, Fauzan A. Teacher Pre-knowledge About Realistic Mathematics Education: A Survey on Teachers in KKG Gugus 5 Bayang, Pesisir Selatan. AIP Conference Proceedings. 2023. p. 1–6.

21. Fauzan A, Yanti WT. The Role of Mathematical Cognition-based Learning Design in Improving a Slow Learner Understanding About Numbers. AIP Conference Proceedings. 2023. p. 1–11.

22. Susanti M, Suyanto, Jailani, Retnawati H. Problem-based Learning for Improving Problem-solving and Critical Thinking Skills: A Case on Probability Theory Course. J Educ Learn. 2023;17(4):507–25.

23. Awoniyi FC, Amponsah KD. Positive Attitudes Toward Mathematics among Senior High School Students in Cape Coast Metropolis. J Educ Learn. 2023;17(2):183–94.

24. Kariman D, Harisman Y, Sovia A, Prahmana RCI. Effectiveness of Guided Discovery-based Module: A Case Study in Padang City, Indonesia. J Math Educ. 2019;10(2):239–50.

25. Suliani M, Juniati D, Lukito A. Mathematics Belief Impact on Metacognition in Solving Geometry: Middle School Students. J Educ Learn. 2024;18(2):286–95.

26. Rifandi R, Tasman F, Rahmi YL, Gusteti MU. A Need Analysis of Developing STEM Integrated Student Worksheet in Multivariate Calculus Course. AIP Conference Proceedings. 2023. p. 1–6.

27. Harisman Y, Ashari J, Amam A, Harun L. The Analysis Application of the Working Backwards Strategy in Mathematic Problem Solving for Senior High School Students Between Gender. AIP Conference Proceedings. 2023. p. 1–7.

28. Listiawati E, Juniati D. An APOS Analysis of Student's Understanding of Quadratic Function Graph. Journal of Physics: Conference Series. 2021. p. 1–9.

29. Martínez-Planell R, Trigueros M. Using Cycles of Research in APOS: The Case of Functions of Two Variables. J Math Behav. 2019;55:1–22.

30. Bansilal S, Brijlall D, Trigueros M. An APOS Study on Pre-service Teachers' Understanding of Injections and Surjections. J Math Behav. 2017;48:22–37.

31. Şefik Ö, Dost Ş. The Analysis of the Understanding of the Three-dimensional (Euclidian) Space and the Two-variable Function Concept by University Students. J Math Behav. 2020;57:1–19.

32. Díaz-Berrios T, Martínez-Planell R. High School Student Understanding of Exponential and Logarithmic Functions. J Math Behav. 2022;66:1–20.

33. Harel G. The Learning and Teaching of Linear Algebra: Observations and Generalizations. J Math Behav. 2017;46:69–95.

34. Nagle C, Martínez-Planell R, Moore-Russo D. Using APOS Theory as a Framework for Considering Slope Understanding. J Math Behav. 2019;54:1-14.

35. Baye MG, Ayele MA, Wondimuneh TE. Implementing GeoGebra Integrated with Multi-teaching Approaches Guided by the APOS Theory to Enhance Students' Conceptual Understanding of Limit in Ethiopian Universities. Heliyon. 2021;7:1–13.

36. Martínez-Planell R, Trigueros M. Students-Understanding of Riemann Sums for Integrals of Functions of Two Variables. J Math Behav. 2020;59:1-26.

37. Borji V, Martínez-Planell R. On Students-Understanding of Volumes of Solids of Revolution: An APOS Analysis. J Math Behav. 2023;70:1-20.

38. Figueroa AP, Possani E, Trigueros M. Matrix Multiplication and Transformations: An APOS Approach. J Math Behav. 2017;52:77–91.

39. García-Martínez I, Parraguez M. The Basis Step in the Construction of the Principle of Mathematical Induction Based on APOS Theory. J Math Behav. 2017;46:128–43.

40. Umam K, Susandi D. Critical Thinking Skills: Error Identifications on Students' with APOS Theory. Int J Eval Res Educ. 2022;11(1):182–92.

41. Tatira B. Mathematics Education Students' Understanding of Binomial Series Expansion Based on the APOS Theory. Eurasia J Math Sci Technol Educ. 2021;17(12):1–13.

42. Arnawa IM, Yerizon, Nita S, Putra RT. Development of Students' Worksheet Based on APOS Theory Approach to Improve Student Achievment in Learning System of Linear Equations. Int J Sci Technol Res. 2019;8(4):287–92.

43. Nagle C, Martínez-Planell R, Moore-Russo D. Using APOS theory as a framework for considering slope understanding. J Math Behav. 2019;54:1-14.

44. Arnawa IM, Yanita, Yerizon, Ginting B, Nita S. Does the use of APOS theory promote students' achievement in elementary linear algebra? Int J Instr. 2021;14(3):175–86.

FINANCING

The authors would like to thank Lembaga Penelitian dan Pengabdian Masyarakat Universitas Negeri Padang for funding this work with a contract number: 1704/UN35.15/LT/2024 fiscal year 2024.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

AUTHORSHIP CONTRIBUTION

Conceptualization: Yerizon Yerizon.
Data curation: Fridgo Tasman.
Formal analysis: Made Arnawa.
Research: Kelly Angelly Hevardani.
Methodology: Yarman UNP.
Project management: Arnellis Arnellis.
Resources: Yerizon Yerizon.
Software: Fridgo Tasman.
Supervision: Made Arnawa.
Validation: Yarman UNP.
Display: Kelly Angelly Hevardani.
Drafting - original draft: Yerizon Yerizon.
Writing - proofreading and editing: Arnellis Arnellis.